Strengthening Quality Mathematics and Science Learning for All: Measuring What Matters

2001 Southwest Regional Conference
National Science Foundation
Division of Educational System Reform

12/14/01
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"The Algebra Issue"

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When Julio asked me to talk about, "The Algebra Issue," I wondered what issue he meant. There are lots of issues around algebra, as the people in this room well know. There's the "who" issue. Who should take algebra? And "why"? And I'll speak to both of those a little bit. I'd actually like to spend more time on the "when" question, which I think is under-discussed right now in our algebra conversations. And then there is the one question that many times we feel like we already have settled—and therefore we move on quickly to making assessments and designing alignments and planning professional development—the "what" question.. But, in fact, I'm going to try to make an argument that we need to keep coming back to the "what" question. What do we mean by algebra?

Let's go back to the who question. The mantra of Algebra for All is pretty well established. It's been taken up in lots of ways and it's evident in lots of places, and I think we need to take this, or at least I take this, as a given. That all students need access to, and good experiences with algebra. And there are a few current trends that I think support this notion and that indicate that this is taking hold. And many of you all come from states and districts that have some kind of a view of Algebra for All. We see it by eighth-grade requirements of algebra. We see it in eighth-grade assessments that include algebra, high school exit exams, university exit exams. Michigan State University actually has not an exam but an exit requirement that students have a certain level of proficiency in algebra. You also see a reason why everyone needs to study algebra in the way that curriculum has chosen to integrate algebra throughout other subjects. So the curricular approaches to data and statistics, for example, have a heavy algebra emphasis. Geometry really invokes a lot of algebra. And once you see the algebra woven into the other core curricula of the school, then it does make sense to think of it as something that all students need to access and have an opportunity to learn. I mentioned the emphasis in the eighth-grade assessments. The role of technology in algebra teaching and learning is central, and that's very much a trend today. And then this broadened perspective of school algebra, the "what" question that we'll come back to.

NAEP data from 2000 reports that 94 percent of grade 12 students claim to have taken algebra I and 80 percent have taken algebra II—that's an increase. And 24 percent take a course titled algebra before or during grade eight, according to self-report. Now what's significant about this is that we are still asking the question about algebra I as if that's a well-defined and well understood identity—or entity. Likewise, with algebra II and I'd like to come back to that.

For 2004, the post-NAEP framework which would address algebra in some way at grades four, eight, and 12 does offer several pieces of the algebra framework that are worth mentioning. Patterns, relations, and functions which come up over and over in any current formulation of goals or standards for algebra, then algebraic representation which is more in the symbolic structural area. Variables, expressions, and operations. And then equations and equality. So it's a heavy dose on the symbol side and yet it still does feature the patterns, relations, and functions side of algebra that starts to show up. ACHIEVE in their work at middle grades and eighth grade also offers expectations in algebra. And functions does show up there as a theme. The TIMSS 2003 assessment framework for algebra also brings forward the notion of patterns and relationships. So when these major national and international assessment kinds of frameworks are featuring a range of algebra topics at several grade levels, there's another argument really why all students need to have the opportunity to be part of algebra.

Now the "why." Algebra's useful and it's powerful. And I didn't prepare a whole lot about this, but in talking with Stephen Marble on the way in, he indicated that there's been some discussion about the need to have strong arguments in one's own community for why everyone needs an experience with algebra. A little bit of that argument depends on the "what." It depends on what kind of algebra we're talking about. And just for your reference, the kind that I am talking about is a fairly broadly based algebra, an algebra that emphasizes function and relationship as well as mathematical structure. Certainly algebra is a gatekeeper to postsecondary education. Placement tests, SATs, ACTs have features that are measuring algebraic skills and reasoning. And for students to place well or to even enter the postsecondary world, algebra's important. Algebra is also a prerequisite for a large number of basic workplace kinds of functions, and it's important to recognize that as the world becomes more technological, some of the reasoning and problem solving that algebra demands can be argued to be applicable in a variety of workplace settings. So although the workplace might not require that one be able to solve equations, it is quite likely to require that one can work with symbols, that one can understand representations. I can't figure out the Xerox machine in our office because it has all of these icons and menus and so forth—a representational system that is highly compacted in terms of what's carried in those symbols very similar to the way in which algebra symbols carry meaning. As well as the more standard spreadsheet and other kinds of computer applications.

Another reason that algebra is quite central and necessary in my view for everyone to have access to, is that it provides the symbol system and mechanisms for working with those symbols that undergird much of mathematics. And so algebra is not separate from geometry or from statistics or from measurement; it actually gives us the symbols and the language for working in those other domains. And in that sense, it serves as an integrating function; it cuts across mathematics, which is a nice feature. And it's also just pretty remarkable that we have it. I mean, I was thinking—you'll probably think this is odd; it's the kind of thing I try to say to my own children—just think of the equation Y = X. Think of how much information is carried in those three very straightforward symbols. I mean, that equation carries meaning that signifies a higher set of points that lie in a very particular way in the Cartesian plane on a line that works, that is placed in a particular direction. The symbol system of algebra is highly compressed and carries enormous information with it, and that's pretty remarkable for the world to understand about. Now I know that those sorts of arguments aren't necessarily always the most popular or the most compelling for parents, but I still think they're good.

So speaking of my own kids, this is my daughter's work in second grade, how many ways can you make 10. And I would argue that this is algebraic as an assignment and also as a teacher working to interpret what she wrote. There are a lot of algebra issues that come up. So you see nine plus one up there at the top. Nine plus one equals 10. And then a little later, one plus nine equals 10. So her—then what would he have been?—seventh- or eighth-grade brother was looking at this and he said, "Well, Adriana, you don't have two different ways there for nine plus one and the one plus nine. That's really the same. So you don't get to count those twice." So a nice little mathematical conversation. Are they or are they not the same? And then when she got onto this zero thing, that took up a whole dinner in our house, trying to talk about whether those were different or not because she was pretty excited and she thought that might mean she had infinitely many ways to make 10. Well, so that's a second grader. But here's the same assignment from the work of Deborah Ball writing number sentences for 10 with third graders. When the kids come upon the following, "We would take any number. It wouldn't matter what number. Say, 200. And then we would minus 200. Okay, got it? Then we would plus 10. and it would always equal 10. Since numbers, they never stop, you could go on and on and on." Point is, at quite young ages, the generalization properties of algebra and the opportunities to do abstraction are fairly wonderful. And Deborah went on to work with those children to try to write the expression in a general form so that they could have a notion of how to express what they had done. So it's a compelling area as well and can be addressed early.

And that brings me to the next question, the "when." And this is one of those two areas that I think are quite under-discussed. When? Well, the National Council of Teachers of Mathematics (NCTM) standards and several other sets of standards recommend that we begin to think about algebra as a pre-K through 12 endeavor. And that's one of the most significant attentions that I think I see with these emphases on the algebra I course in the eighth grade or the ninth grade or on the algebra items on the eighth-grade assessment. Because all of that can look quite different if we begin to take seriously this pre-K through 12 idea of algebra. So Jim Cabot who does research about algebra, talks about algebra before acne and discusses the roots of algebra in young children's thinking. And I showed you a little bit of that with the part, the sums to 10 exercises. There's quite a natural inclination to work in ways that can be generalized and that can be expressed with the language of algebra, even at very early ages. And there's no reason not to pick up on that. There is really no reason why elementary teachers can't be trying to find any algebra in the number work that children do and figuring a way to bring that out and to make sense of it. Cabot also argues that students need to have repeated use of the language of algebra over time in order to become fluent. It's not something that you can meet a little bit in the seventh grade, a little bit more in the eighth grade, and then suddenly be able to work fluently with it. As I started to mention, important mathematical ideas can be pulled together through algebra with the generalization of arithmetic that was pointed to again in the sums to 10, the notion that we have commutativity and that's a key mathematical theme. And the idea of equivalence, which is an algebraic idea that arises in all parts of mathematics.

The high school algebra curriculum is growing broader and, therefore, to be ready for that, it feels quite logical that we could argue for algebra to be included across K-8. There's a lot to attend to in algebra just to prepare students for the high school curriculum. And as I said, kids will do algebraic things. They will discover commutativity themselves, they will find strategies for doing numerical calculations that have nice algebraic features to them. So when they're adding near doubles, you know, if they're adding eight and nine and they know eight plus eight is 16 and then they add on the one to get the 17. There's an algebraic feature to that that has to do with associativity. Right? You're regrouping or rethinking how you state the nine to combine it more readily with the eight. So those algebraic moves that young children do can be brought out and teachers can build on and promote that thinking. So, again, another observation in the Mundy household. My daughter said—again, second grade—when you count by threes, it's odd, then even. So as the teacher trying to think about what she's meaning by that, I'm assuming three is odd and then six is even and then nine is odd and so forth. And so I asked, "What about when you count by twos and by fives?" to see if we could begin to generate a pattern. That was too much at that stage for the second-grade child. The point being by asking those sorts of questions, you could begin to lead toward the notion that we can make general statements about what happens when you skip-count by an odd number, what happens when you skip-count by an even number. And move from something that's very standard in the elementary curriculum to something that just does a little bit of building in the algebra arena. It's an opportunity to generalize.

So I've mentioned that I think that the "when" question is underdeveloped and it can't be taken up really without the "what" question—what is algebra? And do you all know the answer to this? The intensive study of the last three letters of the alphabet. Isn't that how most of us experienced it? It's also probably the way that most of the parents in your communities and the business leaders in your communities experienced it. And it's important to understand that that's where they will be when you raise these issues about we're going to be having this algebra stuff going on and it's going to be done with little kids and so forth. Because you have to work toward a definition of algebra that gets beyond this.

These are some ideas from Hyman Bass who was an eminent mathematician and algebraist. He actually does research and proves theorems in algebra. That's the cutting edge of the field. And his take on school algebra is really that it's about number systems. This is the sort of algebra growing out of arithmetic idea. And it's about the real line and it's about the equations that arise in these systems. So this is a very classical kind of algebra as generalized arithmetic and structure sort of view. This isn't necessarily the whole picture that we're seeing emerge in school mathematics at the high school level these days. And I would characterize that pattern to be more like this, that algebra seems to be shaping itself really as a field that slips over into what we in mathematics would call analysis. It has a very strong functions and relationships emphasis as well as the old structure, X, Y, and Zs, commutativity, and associativity material that we all probably grew up on. The root of both of those parts of algebra could be thought of as the pattern work that you very typically do with young children. So pattern work can take us into these structural features of algebra or into the functions features of algebra which, by the way, another place you need to know that is if you're at a meeting like this because—they took it away but the previous group in this room actually chose to use a function representation to explain one of the ideas that they were trying to convey to the rest of the group, a classic use of the idea of function in a very non-algebraic kind of setting. So these are important ideas that grow out of pattern. And so just to give you a little practice with this, if you haven't been doing any math lately like today, I would urge you to take a look at this and maybe speak to your neighbor about what patterns you see in these two tables. And you can talk to somebody else about that if you'd like.

Okay, so who notices a pattern in the first column? What did you see? Perfect squares. Okay, so did others notice perfect squares in the first column? Yeah. And what about the second column? Can somebody describe what they saw there? The sum of the factors—uh-huh, looking across. I didn't bring a little pointer. So looking here? Two times two equals four and then looking at here at the one times three. So you added the one and the three and you've gotten four, and you've added the two and the two and you've gotten four. And think how much nicer it would be to say that using symbols somehow. We had trouble with the words, just as an aside. But anyhow, what other patterns do you see? Got one here? One we need to hear? Oh, no. Okay. Back here, yeah? Oh, interesting. Seven is the difference here and it's also the difference over here. And then nine, and it's also the difference over here. Nine. But these are always one less. Okay, another observation? Okay, yes, right here. Aha! Did you hear this one? Three plus two—she's looking here and here—gives you the five difference between the three and the eight. Four plus three—looking here and here—gives you the seven difference between the eight and the five. Five plus four gives you the nine difference. You could go the opposite way. Ah, so—here's where overhead projectors were better, you could actually draw those. So, see, four plus one gives you the difference of five. Two plus five gives you the difference of seven, etc. And—ah, over here as well. Two plus three, three plus four. So no shortage of patterns. Okay. So aren't you just dying to know why those patterns hold? But that isn't actually the question I was going to ask. The question I was going to ask was having played with those patterns a little bit, can you multiply

24 by 26 mentally? And what'd you get and why? 624, and why is that? The factors are one up and one down from 25 and it must be the same as 25-squared minus one. And then going in reverse factor, 399. Nineteen times 21 is a way to get yourself started on the 399. You know how those are standard sort of seventh-grade math tasks, find all the factors of some number. And 399 would look ugly, but not if you're thinking 400 because 400 is 20 times 20 and you know from this pattern that that could be sort of translated over into 19 times 21 to get you the 399. And so that's where you get the 19 and the 21. Point being that a very nice and quite straightforward set of patterns that we could approach at quite an early grade would lead you to a set of algebraic calculations that would look very algebra I-ish. It would take you into those calculations in quite a different way. So here's where you would get equations that would look like A-squared equals B and then A minus one times A plus one is equal to B minus one and so forth. Various manipulations you could do. All the patterns that you just observed, you could also try to look at how those would play up with symbols to get a sense of what this is covering.

Okay, I need to say a little bit about NCTM standards and how they take up these questions both of when to do algebra—they recommend it as a K-12 enterprise—and what it should be. In case you're not familiar with this, this book is the revision of NCTM's original 1989 *Curriculum and Evaluation Standards for School Mathematics*. Released in the spring of 2000, it focuses on the classroom-specific curricular parts of that document and some elements of the teaching standards and the assessment standards. And so like the original *Standards* document, it was meant to be a set of goals to guide school mathematics. It's also meant to be a resource for those who make decisions about school mathematics, and a tool to stimulate focus and conversation. What's been interesting about having released this document in the spring of 2000, well after states and districts have developed curricular frameworks, have developed standards, have developed assessments that tie to those, is that we now face an interesting question of how—NCTM faces, actually—how does a document like this, which is coming out after everybody already has some kind of standards—how does this enter the system? In what ways does a document like this get taken up in revision, in rethinking standards? I mean, presumably these systemic change efforts have feedback loops all the way through them that do involve continual looking at standards at some level. We've actually just been engaged, a group of us, in conducting a survey of state supervisors of mathematics to try to understand how our new NCTM standards might or might not play a role in the ongoing revision of state standards documents. So it strikes me that as I'm looking and thinking about this audience, there are folks around the NCTM community who would be very interested in knowing if a document like this is useful to you. But that could be a separate conversation. In any case, if you aren't familiar with it, you might take a look at it on-line at NCTM.org, where it exists in its entirety and where there's also an electronic version that has additional examples and so forth. Basically, like the earlier standards it makes a set of recommendations about what school math ought to look like. And it includes this diagram which is meant to convey that the five subject matter, the five content area standards—number, algebra, geometry, measurement, and data analysis and probability—that those standards need to span the grades. The document is broken into four grade bands if you're interested in the particulars. So there's a pre-K-2 section, a 3-5, 6-8, and 9-12. And the concept here is that number is a theme that should run across the curriculum, pre-K through 12. As you can tell by the color coding here in this representation, what we mean is that it would be a bigger chunk of the pre-K through two curriculum in terms of time than it would be in 9-12, but it is still important in 9-12. It sits next to algebra because of the strong connection to algebra that number has and so forth. And so we don't mean by this that every single area should be sort of treated with equal emphasis in each grade band. But we are making the case in the document that they should span the grades. And the book contains one chapter which is where we lay out what we called in the draft language, the developmental trajectories of each of these areas of mathematics. And we sort of hammered on that language. It was way too stuffy. So that's gone, but the idea is still there. What we have in chapter three of the standards are essays that argue how number ought to grow and be built across the pre-K through 12 years. And how algebra ought to grow and be built across those years. And so those sections, we have heard from some folks that those are useful little synopses for work with principals, with superintendents, with people who are needing some kind of an argument for why you would want to span the grades with these topic.

Each of the standards, then, spans the grades which meant that our grade band groups had to come to agreement about the language because the pre-K people needed to be able to write some specifics at their grade level as did the high school people. And so here is the algebra standard as it sits in the NCTM document today. And you see the four pieces that have been coming up in different places. So this is NCTM's take of the "what" of algebra. Algebra is about understanding patterns, relations, and functions. That this is central to a complete understanding of algebraic ideas. And it can begin with the pattern work that's done in the very earliest of grades.

Secondly, algebra is about representing and analyzing mathematical situations and structures using algebraic symbols. So this is a clear acknowledgement that NCTM supports the idea of developing the symbol systems of algebra. There has been some critique that NCTM hasn't cared about that, but it's quite explicitly in this document and it was in the last one. We are well aware that the compressed symbol system of algebra is a tool like no other mathematics that is crucial for anyone's fundamental work in the field.

The third focus is that we should use mathematical models to represent and understand quantitative relationships. The function that someone presented in the earlier session was a great example. It was sort of a version of a regression equation that showed the various factors that would influence an outcome. And a mathematical model that captures some important aspects of the real world is the sort of thing that algebra should cover across the grades in a way that makes this kind of notion useful for kids.

And then, finally, the notion of analyzing change in various contexts. That's part of the algebra curriculum in the NCTM formulation. So this standard is discussed both in the chapter where I talked about developmental trajectories being described, but then grade band by grade band, the document takes up what the pieces of the standard should be like. So there is the first part—understand patterns, relations, and functions. And here's a sample of what's expressed relative to that topic for pre-K-2. So we go more finely grained into what we call expectations underneath each standard. And if you're not familiar with principles and standards, there's a table at the back of the book that includes all of these for all of the five content standards. And I would suggest that if you are in a position of reevaluating and looking at your frameworks or your standards in whatever your local or regional setting is, that you might want to use this as another kind of resource to consult.

So what does it mean to understand patterns, relations, and functions in pre-K-2? And you see what we argued that it means. It should mean recognizing, describing, and extending. And then by 3-5, we're making generalizations, probably in words more than in symbols. But by 6-8, we're moving then toward the more traditional tools of algebra—tables, graphs, words, and, when possible, symbolic rules. So the document tries to argue how you could go ahead and build those trajectories across grades. And then by high school years, we're talking about both explicit expression of functions and also recursively defined functions.

I'd like to actually go next to another math problem, and this is in the standards in sort of this version. It's also lots of other places. Basically, what I propose here is that you consider the idea of this pattern. And is it clear from where you sit what the pattern is? There's a square that serves as the center. And then around that, a bunch of other squares are attached. And so the first term you see has a single blue square in the center and has how many other squares around it? Eight. Are we okay? All right. And then the second one has four in the middle and more than eight around the edge, and so forth. And something that I have used—and we won't do it in its full glory here, but it has worked fairly well in groups of teachers who span K-12. So I offer it more as a sample of a kind of professional development starter that might be useful in some settings. And here's what I've asked those mixed groups of teachers to do. Very open-ended. Just explore the mathematics that is suggested here. Keep track of your explorations and any conclusions or conjectures that you might make. And so if you do it actually with tiles or cut-out squares, it's most effective because people are drawn toward—what do you suppose? What do they do if they've got tiles on the table? They build these, they make the first one and the second one, and then they make the one that isn't there.

And they don't always do that easily, actually. And then begin to have conversations about what's going on and so forth. But the next question I would ask is for the grade level that you teach, and sketch a lesson plan is a little extreme, but probably talk through some ideas about how you might make a task out of this in class as a way to promote algebraic reasoning, which pushes the teachers in the audience to have to think really about what they would mean by algebraic reasoning. Because if you think algebra is the last three letters of the alphabet, this one is kind of difficult because it doesn't have any letters in it at all yet. So that would be kind of the next thing, and teachers of early childhood will talk about just building these and counting and giving names to some of the suggested concepts and so forth. And so you could extend where's the algebra, find an algebraic expression for the number of tiles in the nth design, explain in words how you found it, and so forth. I'm curious, as you've been looking at it, if anybody has actually tried to generalize this. Has anybody actually gone so far as to maybe make a table that takes a look at what's going on in this problem? Well, it is late in the day. But I would suggest that you might—a nice table to try to make is this one, where you actually create a chart that looks at the number of the square that you're on. So the first square had, remember, one blue in the middle and it had eight around the board. And it used up a total of nine squares. And then the second square, if we went back and looked at it, had two. It was the second square. It had four in the middle, it had 12 around the border, and a total of 16, and so forth and so on. And if you were working with, say, high school kids, you could imagine perhaps getting rather nicely into some algebraic expression. If we're on the nth square, how many blues are there for the nth square? And you might need to play with that a little. On the first square, there was one blue. On the second square, there were —. Third square, there were nine. Fifth square, there would have been —. Nth square, there would be n-squared. Okay? So you might be able to get to that with eighth graders or high school kids who have had some experience working in these kinds of tables. It gets even more interesting when you start to try to relate the number in the border to the number of the square. Maybe we won't do that one together, but I would leave you with that. And I've got a nomination for an answer of four-n plus four. Let's go back and look at the actual picture. Four-n plus four is actually quite beautifully suggested by the picture. Let's look at the second—let's look at the third one, okay? The n for the third square is three. So if you look down this side, you're picking up—the n-square is here, out here in this little part of the border. And you pick up n more here. And n more here and n more here. Here are your four n. And the plus four picks up the corners. But that isn't the way that all kids would think about that. Some might do four times the quantity and plus one. Or any number of other kinds of configurations that lets you talk about associativity and commutativity. It's quite rich, you'll find. You can move it into a graphing calculator exercise. When does the number of squares on the—does the number of squares on the inside ever overtake and get bigger than the number on the border? For example, could that ever happen? In any case, point being there are a lot of contexts—this is just one—that lend themselves pretty well to algebra exploration and across-the-grade algebra exploration.

I just would like to end, then, with a couple of thoughts about the challenge in all of this. I recognize that a lot of this is very far from the nitty-gritty day-to-day issues that you face with your administrative colleagues, with families, with community members, with kids who are up against these high-stakes assessments in algebra. And the problem with this algebra business is that it, too, is a systemic problem. It needs to be treated in the same way that you'd treat your much more global systemic challenges. It involves a very strong need for teacher expertise. Elementary school teachers are generally not prepared to view the arithmetic of the elementary school as being algebraic. And so unless they have some background that helps them see the algebra in what children do and bring it out and start to attend to it, or unless the materials they're using do this explicitly, it's pretty hard to make the early progress that needs to be made. Instructional materials vary enormously with what they do with algebra as a pre-K through 12 process. And being very clear about how they treat algebra, it seems to me, with teachers, with administrators, with parents is a first step in understanding how much progress you're likely to make with the algebra. Getting community and parent support, helping them understand the value and importance of this for all students in the context of a meaningful algebra, not an algebra that's the last three letters of the alphabet, and then making sure that the high-stakes assessments are well aligned with a view of algebra that is a contemporary and comprehensive view, all seem to be quite crucial.

Finally, I think, in a more theoretical sort of way, as a community we still need to get a clearer sense of what we mean by algebraic thinking, what we mean by the "what." I've given you NCTM's view. There are other views. That needs to remain on the table as a topic of conversation among leaders, among classroom teachers who are working with algebra, among those who develop instructional materials. I think, also, although I've given a couple of glimmers, that's hardly a full-blown professional discussion of finding and building on the algebra in the elementary curriculum, and that has to do with the "when." How can we bring this out in some way that systematically develops over time? Which takes me to the next one. How can the ideas of algebra develop in some sensible way and not a random way? We're pretty familiar with using boxes for unknowns in elementary mathematics textbooks, right? Three plus box equals eight. That might be a way to develop the concept of variable. But what about all the other key concepts of algebra like function, like equation? How do we make sure that they grow in some way that's developmentally sensible and that builds? And that was it.. Thank you for your attention at this late time of the day. Thank you.